On the random order extension property on groups

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Orders on groups

Definition

X a set. An order $\prec$ is a binary relation on X s.t.:

1. *$x \prec y$ implies not $y \prec x$;*
2. *$x \prec y$ and $y \prec z$ implies $x \prec z$.*

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Let G be a countable group. G acts on pOrd(G) :

$$a(g \prec)b \Leftrightarrow ag \prec bg.$$

this is called R-action (but it is a left G-action), there is also an L-action

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Theorem (Rhemtulla-Formanek, early 70's)

Torsion-free nilpotent groups have the invariant order extension property.

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No longer true even for metabelian!

IRO-extension property

Let be X a set. Denote $\mathrm{OrdExt}(X) \subset \mathrm{pOrd}(X) \times \mathrm{tOrd}(X)$ the set of all pairs (ω, ω') s.t. $\omega \in \mathrm{pOrd}(X)$, $\omega' \in \mathrm{tOrd}(X)$ and $\omega \subset \omega'$(ω' extends ω).

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A group G has the IRO-extension property iff for every invariant ν on $\mathrm{pOrd}(G)$ there is an invariant γ on $\mathrm{OrdExt}(X)$ s.t. $\mathrm{proj}_{\mathrm{pOrd}(G)}(\gamma) = \nu$.

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A general question: lifting invariant measures over topological extensions:

$$\begin{array}{c} G \curvearrowright X \\ \downarrow \\ G \curvearrowright Y \end{array}$$

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A general question: lifting invariant measures over topological extensions:

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Possible for all extension pairs iff G is amenable.

Partial results

Theorem (A. - Meyerovitch - Ryu 20', Stepin? 70's)

Amenable groups have the IRO extension property.

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$\mathrm{SL}_3(\mathbb{Z})$ *does NOT have the IRO extension property.*

Counterexample: semigroup of matrices with non-negative entries generates a partial invariant order, significantly reworked argument by Witte-Morris 94'.

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Explicit set of counterexamples for the lifting problem:

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Idea of proof

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if $G' < G$ and G has the IRO extension property then so does G'.
Idea:

- maybe F_2 has no IRO extension property?
- ~~each non-amenable group contains F_2~~ [Olshanski, early 80's].

Equivalence relations

Definition

(X, μ) a standard probabilty space. E is a countable Borel equivalence relation:

- *E is a Borel subset of $X \times X$;*
- *E is an equivalence relation;*
- *equivalence classes of E are at most countable.*

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Main example - orbit equivalence relations of measure-preserving actions of countable group on a standard probability space:

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Equivalence relations are high-level analogs of groups.

Gaboriau-Lyons theorem

Theorem (Gaboriau-Lyons 09')

Let G be a non-amenable group. There is an essentially free pmp action of G with orbit equivalence relation E_2 and an essentially free pmp action of F_2 on the same standard probability space with orbit equivalence relation E_1 s.t. $E_1 \subset E_2$.

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Some applications:

- Dixmier problem for lamplighters over non-amenable groups [Monod-Ozawa 09'] ;

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- Ulam non-stability for lamplighters over non-amenable groups [A.22'] .

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1. $f(x) \in M(\mathrm{pOrd}([x]_E))$ *for all* $x \in X$;
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1. $t(x) \in M(\mathrm{OrdExt}([x]_E))$;
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3. $\mathrm{proj}_{\mathrm{pOrd}([x]_E)}(t(x)) = f(x)$ *for a.e.* $x \in X$.

Induction for equiv. rel.

Lemma

Let $E_1 \subset E_2$ be two equivalence relations. If E_2 has the IRO extension property then E_1 has the IRO extension property.

Proof.

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Restrict $t_2(x)$ to $[x]_{E_1}$ for each x to get t for f. □

Extension property for groups and equiv. rel.

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Let ν be a measure on $\mathrm{pOrd}(G)$.

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(and so $M(\mathrm{pOrd}(G))$ with $M(\mathrm{pOrd}([x]_E))$). So we get an IRO f.
Apply the extension property for E to f.

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Apply the extension property for E to f.
get an invariant measure on $X \times \mathrm{pOrd}(G) \times \mathrm{tOrd}(G)$. □

proof continued

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IRO extension property for G implies that for E.

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Idea:

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Idea:

- IRO on E gives a joining of $G \curvearrowright (X, \mu)$ with $G \curvearrowright \mathrm{pOrd}(G)$.
- project to $\mathrm{pOrd}(G)$.

proof continued

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IRO extension property for G implies that for E.

Idea:

- IRO on E gives a joining of $G \curvearrowright (X, \mu)$ with $G \curvearrowright \mathrm{pOrd}(G)$.
- project to $\mathrm{pOrd}(G)$.
- apply the extension property for G

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- relatively independent toining of $G \curvearrowright X \times \mathrm{pOrd}(G)$ and $G \curvearrowright \mathrm{pOrd}(G) \times \mathrm{tOrd}(G)$ over the common factor $\mathrm{pOrd}(G)$.

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- decompose over X.

□

Counterexample for F_2

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$\pi : F_2 \to \mathrm{SL}_3(\mathbb{Z})$, lift over projection.

For $a, b \in G$ denote:

$$\begin{aligned} \mathrm{sml}^{+}_{\sqsubset}(a,b) = &\quad \{\prec \in \mathrm{Ext}(\sqsubset) \,|\, \exists q > 0\, \forall n > 0 \quad a^{-q} b^{n} \prec e\} \\ \mathrm{sml}^{-}_{\sqsubset}(a,b) = &\quad \{\prec \in \mathrm{Ext}(\sqsubset) \,|\, \exists q > 0\, \forall n > 0 \quad e \prec b^{-n} a^{q}\} \end{aligned}$$

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\end{aligned}
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a_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad a_2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad a_3 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$

$$
a_4 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad a_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad a_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.
$$

Denote $\mathrm{sml}_{\sqsubset}^{-} = \bigcap_{i=1}^{6} \mathrm{sml}_{\sqsubset}^{-}(a_i, a_{i-1})$ and $\mathrm{sml}_{\sqsubset}^{+} = \bigcap_{i=1}^{6} \mathrm{sml}_{\sqsubset}^{+}(a_i, a_{i+1})$.

Denote $\mathsf{sml}^-_{\sqsubset} = \bigcap_{i=1}^{6} \mathsf{sml}^-_{\sqsubset}(a_i, a_{i-1})$ and $\mathsf{sml}^+_{\sqsubset} = \bigcap_{i=1}^{6} \mathsf{sml}^+_{\sqsubset}(a_i, a_{i+1})$.

Lemma (GLM22)

$$\mathrm{Ext}(\sqsubset) = \mathsf{sml}^+_{\sqsubset} \cup \mathsf{sml}^-_{\sqsubset}.$$

Let F be a free group and let $\pi : F \to \Gamma$ be an epimorphism. A transversal is any map φ from Γ to F such that $\pi \circ \phi$ is the identity map on Γ.

Fix any $\alpha_1, \ldots, \alpha_6 \in F$ such that $\pi(\alpha_i) = a_i$. Define $\varphi(a_i^n a_{i+1}^m) = \alpha_i^n \alpha_{i+1}^m$, for $i = 1, \ldots, 6 \mod 6$, and $n, m \in \mathbb{Z}$; we define φ on remaining elements of Γ arbitrarily to get a transversal.

Thanks!